Int. J. Heat Mass Transfer. Vol. 10, pp. 1373–1387. Pergamon Press Ltd. 1967. Printed in Great Britain

A LUMPED HEAT-TRANSFER COEFFICIENT FOR PERIODICALLY HEATED HOLLOW CYLINDERS

P. RAZELOS† and A. LAZARIDIS‡

Heat and Mass Transfer Laboratory, Department of Mechanical Engineering, Columbia University, New York, N.Y. 10027

(*Received* 10 *November* 1966 *and in revised form* 8 *May* 1967)

Abstract—A lumped heat-transfer coefficient has been developed for periodically heated and subsequently cooled hollow cylinders of infinite extent. This corrected heat-transfer coefficient incorporates the effect of the wall resistance and preserves the heat transferred per period. The treatment has been generalized by considering the heating or cooling to take place at the inside or at the outside surface of the hollow cylinder. To evaluate the lumped heat-transfer coefficient, all the necessary time-mean temperatures have been determined analytically. The subsequent expressions for numerical answers were evaluated on a IBM 7094 digital computer. The results are presented graphically in the form of a correction factor for the heat-transfer coefficient. Finally, an example is presented to show the practical application of the results.

NOMENCLATURE

A, constant of integration;

A_n, constant of integration of the nth term of the solution defined by equations (A.12), (A.20a, b), and (A.20);

B, constant of integration;

B_n, constant of integration of the nth term of the solution during the cooling period, defined by equations (A.20a, b) and (A.20);

b, constant of integration defined by equation (A.7b);

C, constant of integration;

C_s, specific heat of conducting medium [Btu/lb °F];

c, constant of integration defined by equation (A.7c);

F, function of the variable defined by equation (5a);

F_0, constant representing heat flux imposed at the boundary $r = r_1$ [Btu/h];

f, function defined by equation (A.14c);

G_n, function of Bessel functions defined by equation (A.12a);

H, constant defined by equation (B.3a);

h, heat-transfer coefficient [Btu/h ft^2 °F];

h^*, lumped heat-transfer coefficient [Btu/h ft^2 °F];

K, constant of integration defined by equation (B.3b);

K_n, factor related with the nth term of the solution and defined by equation (A.20);

k, conductivity of conducting medium [Btu/h ft^2 °F/ft];

L, constant of integration appearing in equation (B.5);

l, volume of cylindrical wall per unit surface area [ft];

Q_n, function defined by equation (5c);

r, radius;

r_1, radius of heating or cooling surface [ft];

† Director, Heat and Mass Transfer Laboratory.

‡ Research Engineer, Heat and Mass Transfer Laboratory.

r_2, radius of insulated surface [ft];
T, temperature [°F];
T_m, space-mean temperature of conducting medium [°F];
T_s, temperature of hollow cylinder at the heat-transfer surface, $r = r_1$ [°F];
T_a, bulk temperature of fluid in contact with the hollow cylinder at the heat-transfer surface [°F];
T_1, T_2, two different solutions of T defined by equations (A.7) and (A.12);
s, parameter of Laplace transformation;
t, time [h];
x, variable defined by equation (B.14a);
y, function of x defined by equation (B.14b);
y', first derivative of y, (dy/dx);
Z, function defined by equations (A.9) and (A.10a).

Greek symbols
α, thermal diffusivity;
β, ratio of r_2 to r_1;
γ, constant from [4] also defined by equation (B.1);
δ, radial wall thickness of hollow cylinder [ft];
ζ, dummy variable corresponding to τ;
Θ, function of τ in the separation of variables technique;
λ, special decay constant from [4];
μ, ratio of heating and cooling periods, (τ_h/τ_c);
ξ, ratio of radius to radial wall thickness (r/δ);
Π, harmonic mean period, equal to $2/(1/\tau_c + 1/\tau_h)$;
ρ, mass density of conducting medium;
σ, constant from [4], also defined by equation (B.1);
τ, dimensionless time, $(\alpha t/\delta^2)$;
τ_h, dimensionless time span for heating;
τ_c, dimensionless time span for cooling;
Φ, correction factor;
χ, function defined by equation (5b).

Primed quantities refer to values of the respective symbols during the heating period. The same symbols represent the cooling period if they are double-primed. Underlined letters represent Laplace Transforms of the respective symbols.

INTRODUCTION

In certain engineering problems involving periodic heat-transfer processes, the solution of the differential equations describing the performance of the system can be made easier by means of a lumped heat-transfer coefficient. The correction factor used for this purpose accounts also for the effect of the wall resistance besides the fluid resistance. Hausen, in his studies on regenerators, developed a method to determine such a correction factor corresponding to the three cases of periodically heated and cooled slabs, circular cylinders and spheres [1]. Recently, Butterfield *et al.* [2] developed a correction factor in a graphical form by solving the pertinent differential equations by the finite difference technique. The cases they treated were those for a hollow square section, a hollow cylinder and a slab.

In the present paper a lumped heat-transfer coefficient is developed for an infinitely long hollow cylinder and a correction factor is determined therefrom. The cylinder is being heated periodically at one surface while the other surface is insulated. The method employed is similar to the analytical method of Hausen. The main assumptions are that the heat-transfer coefficients are constant and that the temperature difference between the fluid temperature and the cylinder wall surface temperature remains unchanged. This second assumption essentially means that the cylinder is heated by a constant heat flux.

The work of this paper was motivated by the necessity of finding a lumped heat-transfer coefficient to use in the design of regenerative heat exchangers having circular checkerwork section. It must be pointed out that the assumption of a constant temperature difference between the flowing gas and the surface of the wall prevails essentially in the middle portion of the regenerator [1].

STATEMENT OF THE PROBLEM

Given an infinitely long hollow cylinder of circular cross-section, with constant properties insulated along the periphery at $r = r_2$ and periodically heated and cooled, respectively, by applying a constant heat flux on the surface at $r = r_1$. The variable wall temperature is replaced by its space-mean temperature T_m, (lumped wall). A lumped heat-transfer coefficient h^*, is introduced to account for the wall resistance and to keep the heat transferred per period unchanged. With this postulation, h^* is defined as follows:

$$h^* \int_0^\tau (T_a - T_m)\,d\zeta = h \int_0^\tau (T_a - T_s)\,d\zeta. \quad (1)$$

The time-mean of equation (1) gives

$$h^*(\overline{T}_a - \overline{T}_m) = h(\overline{T}_a - T_s) \quad (2a)$$

or

$$\frac{1}{h^*} = \frac{1}{h}\left[\frac{\overline{T}_a - \overline{T}_m}{\overline{T}_a - \overline{T}_s}\right]. \quad (2b)$$

Equation (2b) is written in a more convenient form below:

$$\frac{1}{h^*} = \frac{1}{h}\left[1 + \frac{h\delta}{3k}\Phi\right] \quad (3)$$

where

$$\Phi = \frac{3k}{h\delta}\left[\frac{\overline{T}_s - \overline{T}_m}{\overline{T}_a - \overline{T}_s}\right]. \quad (3a)$$

It is shown in Appendix A that the correction factor Φ, is a function of the harmonic mean period Π, the ratio of the radii of the insulated surface over the heat transfer surface, and the ratio of the heating and cooling periods; i.e.

$$\Phi = \Phi(\Pi, \beta, \mu). \quad (4)$$

Equation (3a) reveals that Φ can be obtained from a knowledge of T_m and T_s, quantities that can be found directly from the mathematical solution of the problem. This solution is also presented in Appendix A.

PRESENTATION OF RESULTS

The solution presented in Appendix A defines the correction factor Φ, by the equation

$$\Phi = F(\beta) + \chi(\beta, \Pi, \mu) \quad (5)$$

where

$$F(\beta) = \frac{3[\beta^4(4\ln\beta - 3) + 4\beta^2 - 1]}{4(\beta - 1)^3(\beta + 1)^2} \quad (5a)$$

and

$$\chi(\beta, \Pi, \mu) = -\frac{12(\beta - 1)}{\Pi}\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n^4}\right)\left(\frac{1}{\beta^2 Q_n^2 - 1}\right) \frac{\{1 - \exp[-\lambda_n^2\Pi(1 + \mu)/2\mu]\}.\{1 - \exp[-\lambda_n^2\Pi(1 + \mu)/2]\}}{\{1 - \exp[-\lambda_n^2\Pi(1 + \mu)^2/2\mu]\}}. \quad (5b)$$

Here,

$$Q_n = \frac{J_1[\lambda_n/(\beta - 1)].Y_0[\beta\lambda_n/(\beta - 1)] - J_0[\beta\lambda_n/(\beta - 1)].Y_1[\lambda_n/(\beta - 1)]}{J_1[\lambda_n/(\beta - 1)].Y_0[\lambda_n/(\beta - 1)] - J_0[\lambda_n/(\beta - 1)].Y_1[\lambda_n/(\beta - 1)]} \quad (5c)$$

and λ_n are the roots of the equation

$$J_1[\lambda_n/(\beta - 1)]\,Y_1[\beta\lambda_n/(\beta - 1)] - J_1[\beta\lambda_n/(\beta - 1)]\,Y_1[\lambda_n/(\beta - 1)] = 0. \quad (5d)$$

This equation was solved on an IBM 7094 digital computer and the roots, λ_n were determined. Some of the roots were also compared with the values given in [3]. The first five roots of equation (5d) are listed in Table 1.†

Equation (5) was also programmed and solved for Φ on the 7094 computer. The results are presented in graphical form in Figs. 1–6, where Φ is plotted versus $1/\Pi$ with μ and β as parameters. In the limiting case of Π approaching infinity, i.e. $\Pi \to \infty$ or $1/\Pi \to 0$, equation (5) becomes

$$\lim_{\Pi\to\infty} \Phi = F(\beta) \tag{6}$$

since

$$\lim_{\Pi\to\infty} \chi(\beta, \Pi, \mu) = 0.$$

On the other hand, if β approaches the value of one, $F(\beta)$ attains the value of 1 and $\chi(\beta, \Pi, \mu)$ takes the form

$$\lim_{\beta\to 1} \chi(\beta, \Pi, \mu) = -\frac{2}{\Pi}\sum_{n=1}^{\infty}\left[\frac{1}{(n\pi)^4}\right]\left[\frac{\{1-\exp[-n^2\pi^2(1+\mu)\Pi/2\mu]\}}{\{1-\exp[-n^2\pi^2(1+\mu)^2\Pi/2\mu]\}}\right]\{1-\exp[-n^2\pi^2(1+\mu)\Pi/2]\} \tag{7}$$

Thus,

$$\lim_{\beta\to 1} \Phi = 1 - \frac{2}{\Pi}\sum_{n=1}^{\infty}\left[\frac{1}{(n\pi)^4}\right]\frac{\{1-\exp(-[n^2\pi^2(1+\mu)\Pi/2\mu])\}.\{1-\exp(-[n^2\pi^2(1+\mu)\Pi/2])\}}{\{1-\exp(-[n^2\pi^2(1+\mu)^2\Pi/2\mu])\}} \tag{8}$$

Table 1

β	λ_1	λ_2	λ_3	λ_4	λ_5
0·02	−3·75967	−6·89038	−10·00041	−13·24224	−16·21753
0·1	−3·54685	−6·59754	−9·67347	−12·76974	−15·87900
0·2	−3·38860	−6·44432	−9·54128	−12·65684	−15·78164
0·4	−3·23471	−6·33464	−9·45988	−12·59292	−15·72929
0·5	−3·19658	−6·31235	−9·44447	−12·58120	−15·71992
0·6	−3·17204	−6·29891	−9·43533	−12·57431	−15·71432
0·8	−3·14751	−6·28616	−9·42676	−12·56786	−15·70916
1·0	3·15149	6·28319	9·42478	12·56637	15·70796
1·2	2·14555	6·28532	9·42610	12·56737	15·70876
1·4	3·15498	6·28997	9·42926	12·57005	15·71069
1·5	3·16094	6·29306	9·43149	12·57134	15·71225
1·6	3·16746	6·29648	9·43369	12·56540	15·71332
1·8	3·18169	6·30407	9·43882	12·47693	15·71367
2·0	3·19658	6·31235	9·44447	12·58120	15·71992
2·5	3·23471	6·33464	9·45988	12·59292	15·72929
3·0	3·27128	6·35768	9·47618	12·60544	15·73942
3·5	3·30500	6·38052	9·49274	12·61829	15·74987
4·0	3·33563	6·40269	9·50922	12·63122	15·76046

Equation (8) is identical with Hausen's equation for a slab, which is the limiting case when β approaches one.

† The roots for $\beta < 1$ are deduced from the roots for $\beta > 1$ by the equation $\lambda_{n(\beta)} = -\lambda_{n(1/\beta)}$.

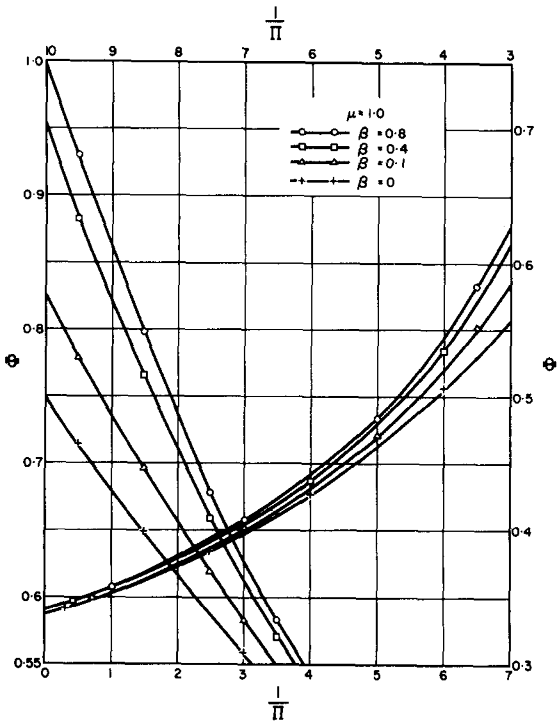

FIG. 1. Correction factor, Φ, for $\mu = 1$ and $\beta = 0{\cdot}8, 0{\cdot}4, 0{\cdot}1, 0$.

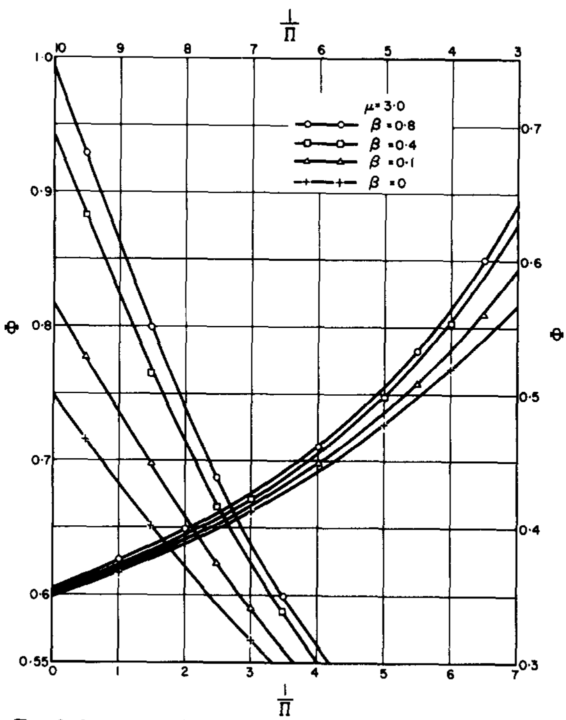

FIG. 2. Correction factor, Φ, for $\mu = 3$ and $\beta = 0{\cdot}8, 0{\cdot}4, 0{\cdot}1, 0$.

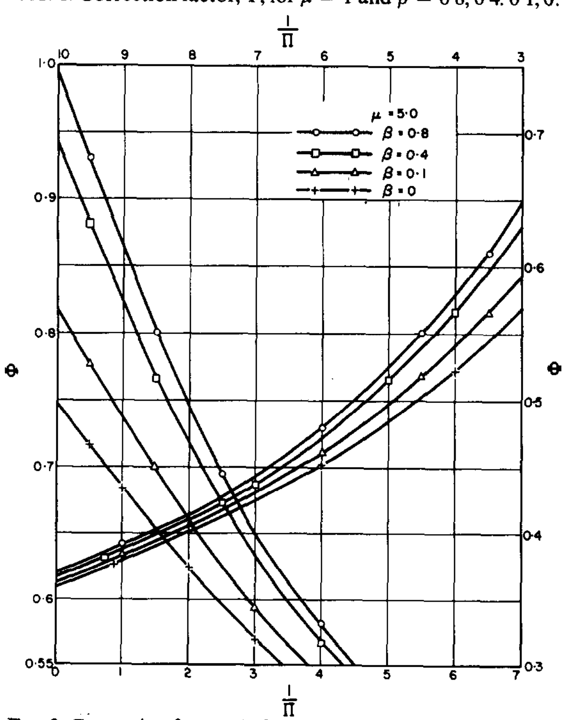

FIG. 3. Correction factor, Φ, for $\mu = 5$ and $\beta = 0{\cdot}8, 0{\cdot}4, 0{\cdot}1, 0$.

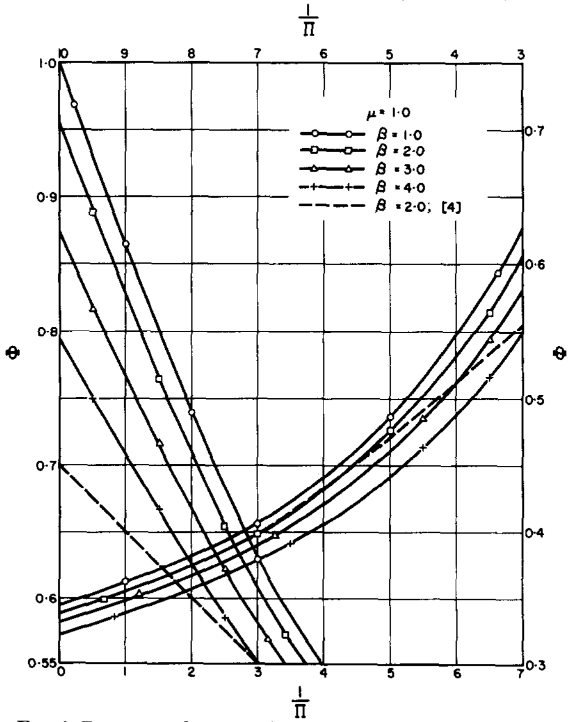

FIG. 4. Correction factor, Φ, for $\mu = 1$ and $\beta = 1{\cdot}0, 2{\cdot}0, 3{\cdot}0, 4{\cdot}0$.

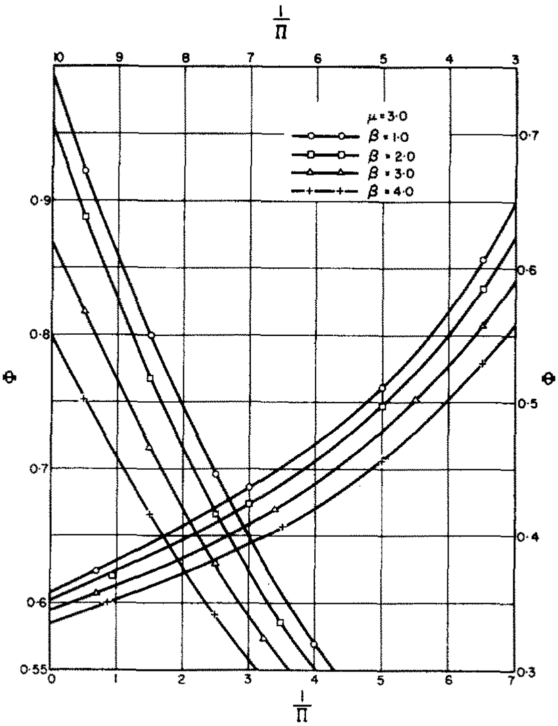

FIG. 5. Correction factor, Φ, for $\mu = 3$ and $\beta = 1{\cdot}0, 2{\cdot}0, 3{\cdot}0, 4{\cdot}0$.

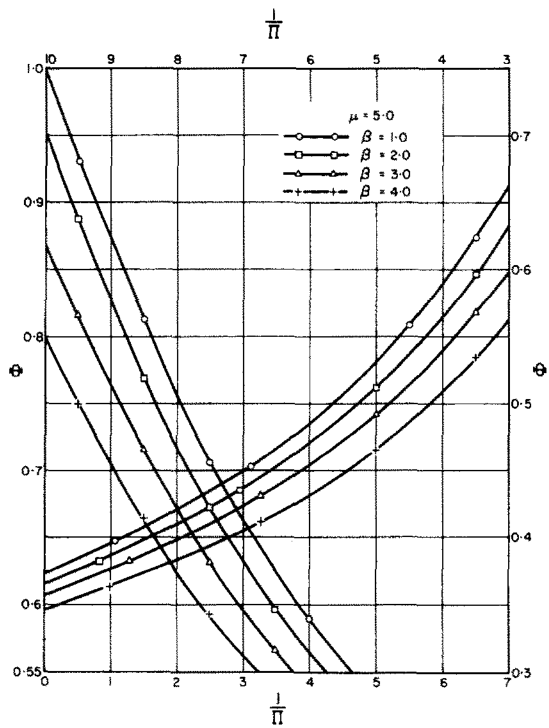

FIG. 6. Correction factor, Φ, for $\mu = 5$ and $\beta = 1{\cdot}0, 2{\cdot}0, 3{\cdot}0, 4{\cdot}0$.

The validity of the results of the present paper for large values of the periods has been proven by the lumped parameter technique of [4] and is presented in Appendix B.

A comparison of the results of this paper with those developed in [2] shows a large discrepancy for all $\Pi > 0{\cdot}25$ (see Fig. 4). It is interesting to observe that the larger the harmonic mean of the periods, the larger the difference between the values of Φ in the two papers. The results of the present paper for large values of Π have been checked by the method of [4], which is presented in Appendix B. The following example will help the reader apply these results.

Example. A cylindrical regenerator of I.D. 0·125 ft and O.D. 0·25 ft is made of superduty fireclay brick with the following properties: $\rho = 100\text{–}140$ lb/ft^3, $c = 0{\cdot}3$ Btu/lb °F, $k = 0{\cdot}9$ Btu/h ft °F, $\alpha = 0{\cdot}9/0{\cdot}3 \times 120 = 0{\cdot}025$ ft^2/h. The heat-transfer coefficients during the heating and cooling periods are 8 and 6 Btu(h ft^2 °F, respectively. For a heating period of 3 h and a cooling period of 1 h

$$\tau_h = \frac{0{\cdot}025 \times 1}{0{\cdot}125} = \frac{8}{5}, \qquad \tau_c = \frac{0{\cdot}025 \times 3}{0{\cdot}125^2} = \frac{24}{5}$$

where $0{\cdot}125 = \delta$ and $\delta = r_2 - r_1$.

$$\frac{1}{\Pi} = \frac{1}{2}\left[\frac{5}{8} + \frac{5}{24}\right] = \frac{5}{12}.$$

Fig. 5, with $\mu = 3$ and $\beta = 2$, gives $\Phi = 0{\cdot}9$.

$$\frac{1}{h^{*\prime}} = \frac{1}{8} + 0{\cdot}9 \times \frac{0{\cdot}125}{3 \times 0{\cdot}9} = \frac{1}{6},$$

$$\frac{1}{h^{*\prime\prime}} = \frac{1}{6} + \frac{0{\cdot}9}{3 \times 0{\cdot}9}\frac{0{\cdot}125}{} = \frac{5}{24}.$$

Hence,

$$h^{*\prime} = 6 \frac{Btu}{\text{h/ft}^2\,°\text{F}}$$

and

$$h^{*\prime\prime} = 4{\cdot}8 \frac{Btu}{\text{h/ft}^2\,°\text{F}}.$$

ACKNOWLEDGEMENT

The authors wish to thank the AISE for their financial support which made this investigation possible. Acknowledgements are also due to the Columbia Computer Center for the assistance provided during the course of the numerical computations.

REFERENCES

1. H. HAUSEN, *Wärmeübertragung im Gegenstrom, Gleichstrom und Kreuzstrom.* Springer-Verlag, Berlin–Göttingen–Heidelberg (1950).
2. P. BUTTERFIELD, J. S. SCHOFIELD and P. A. YOUNG, Hot blast stoves—Part II, *J. Iron Steel Inst.* **201**, 497–508 (1963).
3. M. ABRAMOWITZ and I. A. STEGUN, *Handbook of Mathematical Functions*, U.S. Department of Commerce, National Bureau of Standards Applied Mathematics Series 55 (1965).
4. H. G. ELROD JR., Improved lumped parameter method for transient heat conduction calculations, *J. Heat Transfer* **82**, 181–188 (1960).

APPENDIX A

The differential equation governing the heat flow in the system is

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t}. \tag{A.1}$$

The boundary conditions pertinent to the problem are

$$\frac{\partial T}{\partial r} = 0 \quad \text{at} \quad r = r_2 \tag{A.2a}$$

$$\frac{\partial T}{\partial r} = -\frac{F_0}{k} \quad \text{at} \quad r = r_1. \tag{A.2b}$$

The initial conditions are those of cyclic equilibrium, namely

$$T'(0) = T''(\tau_c) \tag{A.2c}$$

$$T'(\tau_h) = T''(0). \tag{A.2d}$$

Method of solution

Introducing the dimensionless parameters

$$\xi \equiv \frac{r}{\delta} \quad \text{and} \quad \tau \equiv \frac{\alpha t}{\delta^2} \tag{A.3}$$

the above equations become

$$\frac{\partial^2 T}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial T}{\partial \xi} = \frac{\partial T}{\partial \tau} \tag{A.4}$$

$$\frac{\partial T}{\partial \xi} = 0 \quad \text{at} \quad \xi = \xi_2 \tag{A.4a}$$

$$\frac{\partial T}{\partial \xi} = -\frac{F_0\delta}{k} \quad \text{at} \quad \xi = \xi_1. \tag{A.4b}$$

This nonhomogeneous problem can be broken into two simpler ones. Thus, we select

$$T = T_1 + T_2 \tag{A.5}$$

where T_1 must satisfy equations (A.4), (A.4a), and (A.4b) and T_2 must satisfy equations (A.4), (A.4a) and

$$\frac{\partial T_2}{\partial \xi} = 0 \quad \text{at} \quad \xi = \xi_1. \tag{A.6}$$

From equations (A.4), (A.4a), and (A.4b), solved for T_1, one can easily ascertain that

$$T_1 = a\xi^2 + b \ln \xi + c\tau \tag{A.7}$$

where

$$a = (F_0 \delta \xi_1)/[2k(\xi_2^2 - \xi_1^2)] \tag{A.7a}$$

$$b = -(F_0 \delta \xi_1 \xi_2^2)/[k(\xi_2^2 - \xi_1^2)] \tag{A.7b}$$

$$c = (2F_0 \delta \xi_1)/[k(\xi_2^2 - \xi_1^2)]. \tag{A.7c}$$

Meanwhile, equations (A.4), (A.4a) and (A.6), with $T = T_2$, take the form

$$\frac{\partial^2 T_2}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial T_2}{\partial \xi} = \frac{\partial T_2}{\partial \tau} \tag{A.8}$$

$$\frac{\partial T_2}{\partial \xi} = 0 \quad \text{at} \quad \xi = \xi_2 \tag{A.8a}$$

$$\frac{\partial T_2}{\partial \xi} = 0 \quad \text{at} \quad \xi = \xi_1. \tag{A.8b}$$

Separation of variables with

$$T_2 = Z(\xi)\Theta(\tau) \tag{A.9}$$

renders the solutions

$$Z(\xi) = AJ_0(\lambda\xi) + BY_0(\lambda\xi) \tag{A.10a}$$

and

$$\Theta(\tau) = C \exp(-\lambda^2 \tau) \tag{A.10b}$$

where λ is a constant.

The application of the boundary conditions (A.8a) and (A.8b) on equations (A.10a) gives the following transcendental equation, the solution of which supplies information on the eigenvalues λ_n:

$$J_1(\lambda\xi_1) . Y_1(\lambda\xi_2) - J_1(\lambda\xi_2) . Y_1(\lambda\xi_1) = 0. \tag{A.11a}$$

Also

$$A = -B\frac{Y_1(\lambda_n\xi_2)}{J_1(\lambda_n\xi_2)} = -B\frac{Y_1(\lambda_n\xi_1)}{J_1(\lambda_n\xi_1)}. \tag{A.11b}$$

Thus,

$$T_2 = \sum_{n=1}^{\infty} A_n \exp(-\lambda_n^2 \tau)\, G_n \tag{A.12}$$

where

$$G_n = Y_0(\lambda_n \xi)\, J_1(\lambda_n \xi_1) - J_0(\lambda_n \xi)\, Y_1(\lambda_n \xi_1). \tag{A.12a}$$

Now,

$$T_m = \frac{2}{r_2^2 - r_1^2} \int_{r_1}^{r_2} r T(r)\, \mathrm{d}r \tag{A.13a}$$

or

$$T_m = \frac{2}{\xi_2^2 - \xi_1^2} \int_{\xi_1}^{\xi_2} \xi T(\xi)\, \mathrm{d}\xi. \tag{A.13b}$$

Since

$$\int_{\xi_1}^{\xi_2} \xi T_2(\xi)\, \mathrm{d}\xi = 0,$$

equation (A.13b) gives

$$T_m = c\tau + f(\xi_1, \xi_2) \tag{A.14a}$$

or

$$T_m = T_1 + f(\xi_1, \xi_2) - a\xi^2 - b \ln \xi \tag{A.14b}$$

where

$$f(\xi_1, \xi_2) = \frac{a}{2}(\xi_2^2 + \xi_1^2) + \frac{b}{\xi_2^2 - \xi_1^2}(\xi_2^2 \ln \xi_2 - \xi_1^2 \ln \xi_1) - \frac{b}{2}. \tag{A.14c}$$

Equation (A.14b) can be written for T_1 as follows:

$$T_1 = T_m + a\xi^2 + b \ln \xi - f(\xi_1, \xi_2). \tag{A.15}$$

Thus, the complete solution of the original differential equation for T, under boundary conditions (A.4a) and (A.4b) is

$$T = T_m(\tau) + a\xi^2 + b \ln \xi - f(\xi_1, \xi_2) + \sum_{n=1}^{\infty} A_n \exp(-\lambda_n^2 \tau)\, G_n. \tag{A.16}$$

During the heating and cooling periods, the following equations hold, respectively:

$$T' = T'_m(\tau) + a'\xi^2 + b' \ln \xi - f'(\xi_1, \xi_2) + \sum_{n=1}^{\infty} A_n \exp(-\lambda_n^2 \tau)\, G_n(\lambda_n \xi) \tag{A.17a}$$

$$T'' = T''_m(\tau) + a''\xi^2 + b'' \ln \xi - f''(\xi_1, \xi_2) + \sum_{n=1}^{\infty} B_n \exp(-\lambda_n^2 \tau)\, G_n(\lambda_n \xi). \tag{A.17b}$$

The constants A_n and B_n are then determined from the conditions of cyclic equilibrium:

$$T'(0) = T''(\tau_c) \tag{A.2c}$$

$$T'(\tau_h) = T''(0). \tag{A.2d}$$

Thus,

$$(a' - a'')\,\xi^2 + (b' - b'') \ln \xi + f''(\xi_1, \xi_2) - f'(\xi_1, \xi_2) = \sum_{n=1}^{\infty} (B_n \exp(-\lambda_n^2 \tau_c)\, A_n)\, G_n(\lambda_n \xi) \tag{A.18}$$

$$(a' - a'')\,\xi^2 + (b' - b'') \ln \xi + f''(\xi_1, \xi_2) - f'(\xi_1, \xi_2) = \sum_{n=1}^{\infty} [B_n - A_n \exp(-\lambda_n^2 \tau_h)]\, G_n(\lambda_n \xi). \tag{A.19}$$

The usual orthogonality relations provide the following system of equations for A_n and B_n:

$$B_n \exp(-\lambda_n^2 \tau_c) - A_n = K_n \tag{A.20a}$$

$$B_n - A_n \exp(-\lambda_n^2 \tau_h) = K_n \tag{A.20b}$$

where

$$K_n = \frac{\int_{\xi_1}^{\xi_2} [(a' - a'')\,\xi^3 + (b' - b'')\,\xi \ln \xi]\, G_n(\lambda_n \xi)\, d\xi}{\int_{\xi_1}^{\xi_2} \xi\, G_n^2(\lambda_n \xi)\, d\xi}. \tag{A.20c}$$

Introducing the corresponding equations for a', a'', b', and b'' and using the heat balance relation

$$F'_0 \tau_h = F''_0 \tau_c \tag{A.20d}$$

the following differences can easily be obtained:

$$a' - a'' = \left[\frac{F'_0 \delta \xi_1}{2(\xi_2^2 - \xi_1^2)\, k}\right] \cdot \left[\frac{\tau_h + \tau_c}{\tau_c}\right] \tag{A.20e}$$

$$b' - b'' = -\left[\frac{F'_0 \delta \xi_1 \xi_2^2}{k(\xi_2^2 - \xi_1^2)}\right] \cdot \left[\frac{\tau_h + \tau_c}{\tau_c}\right]. \tag{A.20f}$$

The integrals to be evaluated in equation (A.20c) are

$$\int_{\xi_1}^{\xi_2} \xi\, G_n^2(\xi)\, d\xi = \tfrac{1}{2}[\xi_2^2\, G_n^2(\xi_2) - \xi_1^2\, G_n^2(\xi_1)] \tag{A.20g}$$

$$\int_{\xi_1}^{\xi_2} \xi^3 G_n(\xi)\,\mathrm{d}\xi = \frac{2}{\lambda_n^2}[\xi_2^2 G_n(\xi_2) - \xi_1^2 G_n(\xi_1)] \tag{A.20h}$$

$$\int_{\xi_1}^{\xi_2} \xi \ln \xi\, G_n(\xi)\,\mathrm{d}\xi = \frac{G_n(\xi_2) - G_n(\xi_1)}{\lambda_n^2}. \tag{A.20i}$$

Thus,

$$K_n = \frac{2(\beta - 1)\,G_n(\xi_1)}{\lambda_n^2[\beta^2 G_n^2(\xi_2) - G_n^2(\xi_1)]}. \tag{A.20}$$

Inserting this value of K_n in equations (A.20a) and (A.20b), A_n and B_n are easily obtained. They, in turn, give $T_s - T_m$.

The time average of the difference $T_s - T_m$, which is required to determine the correction factor, is

$$\overline{T}_s - \overline{T}_m = a'\xi_1^2 + b'\ln \xi_1 - f'(\xi_1, \xi_2)$$
$$-\frac{1}{\tau_h}\sum_{n=1}^{\infty} K_n G_n \frac{[1 - \exp(-\lambda_n^2\tau_c)][1 - \exp(-\lambda_n^2\tau_h)]}{\lambda_n^2\{1 - \exp[-\lambda_n^2(\tau_c + \tau_n)]\}}. \tag{A.21}$$

With equation (A.21) developed and the use of equation (A.14c) and the definition $\beta = \xi_2/\xi_1$, the correction factor Φ, can be obtained in the form

$$\Phi = F(\beta) + \chi(\Pi, \beta, \mu) \tag{A.22}$$

where $F(\beta)$ and $\chi(\Pi, \beta, \mu)$ are given in equations (5a) and (5b), respectively, and Π is the harmonic mean of the periods; i.e.

$$\Pi = \frac{2}{\dfrac{1}{\tau_c} + \dfrac{1}{\tau_h}}. \tag{A.23}$$

When β approaches one, $\chi(\Pi, \beta, \mu)$ attains the form of equation (7), as follows. As β comes closer and closer to one, $[1/(\beta - 1)]$ increases to higher and higher values. The asymptotic values of the Bessel functions for large values of the argument give the following equation in substitution of equation (5d):

$$\cos\left(\frac{\lambda_n}{\beta - 1} - \frac{3\pi}{4}\right) * \sin\left(\frac{\beta\lambda_n}{\beta - 1} - \frac{3\pi}{4}\right) - \cos\left(\frac{\beta\lambda_n}{\beta - 1} - \frac{3\pi}{4}\right)\sin\left(\frac{\lambda_n}{\beta - 1} - \frac{3\pi}{4}\right) = 0 \tag{A.24a}$$

or

$$\sin(\lambda_n) = 0. \tag{A.24b}$$

Hence,

$$\lambda_n = n\pi. \tag{A.24c}$$

In the same manner, one can easily prove that

$$\lim_{\beta\to 1}[6(\beta - 1)]/[\lambda_n^4(\beta^2 Q_n^2 - 1)] \to (n\pi)^{-4}. \tag{A.25}$$

APPENDIX B

The application of the lumped parameter technique of [4] to the present problem gives a check for values of $\Pi \to \infty$. The conditions of [4] are met completely in the present case since the method was used to check the validity of the previously derived results in a region where the periods become very large. The reference gives the equation

$$\frac{\mathrm{d}T_m(\tau)}{\mathrm{d}\tau} + \sigma\, T_m(\tau) = \sigma\, T_s(\tau) + \gamma \frac{\mathrm{d}T_s(\tau)}{\mathrm{d}\tau} \tag{B.1}$$

where

$$\tau = \frac{\alpha t}{\delta^2} \tag{B.1a}$$

and

$$\rho C_s l \frac{\mathrm{d}T_m(t)}{\mathrm{d}t} = h\,(T_a - T_s). \tag{B.1b}$$

In the present case

$$h(T_a - T_s) = k \frac{\delta T}{\partial r} = -F_0. \tag{B.2}$$

Equations (B.1b) and (B.2) give

$$\frac{\mathrm{d}T_m}{\mathrm{d}t} = -\frac{F_0}{\rho C_s l} \equiv H \tag{B.3a}$$

or

$$T_m = Ht + K. \tag{B.3b}$$

Now, from equation (B.1)

$$\frac{\mathrm{d}T_m(t)}{\mathrm{d}t} + \frac{\sigma\alpha}{\delta^2} T_m(t) = \frac{\sigma\alpha}{\delta^2} T_s(t) + \gamma \frac{\mathrm{d}T_s(t)}{\mathrm{d}t}. \tag{B.4}$$

Therefore,

$$T_s(t) = L \exp\left(-\frac{\sigma\alpha}{\gamma\delta^2} t\right) + Ht + H(1-\gamma)\frac{\delta^2}{\sigma\alpha} + K. \tag{B.5}$$

Again from equation (B.1)

$$\bar{T}_s - \bar{T}_m = \frac{1}{\sigma}\left[\frac{\mathrm{d}T_m}{\mathrm{d}\tau} - \frac{\gamma}{\tau}\{T_s(\tau) - T_s(0)\}\right]. \tag{B.6}$$

Applying equation (B.5) to the heating and cooling periods, respectively, and referring to conditions of thermal equilibrium, one obtains

$$L' = (H'' - H')(1-\gamma)\left(\frac{\delta^2}{\sigma\alpha}\right)\left[\frac{1 - \exp(-\sigma\tau_c/\gamma)}{1 - \exp\left(-\frac{\sigma\tau_c}{\gamma} - \frac{\sigma\tau_h}{\gamma}\right)}\right] \tag{B.7a}$$

and

$$-L' = (H'' - H')(1-\gamma)\left(\frac{\delta^2}{\sigma\alpha}\right)\left[\frac{1-\exp(-\sigma\tau_h/\gamma)}{1-\exp\left(-\frac{\sigma\tau_c}{\gamma}-\frac{\sigma\tau_h}{\gamma}\right)}\right]. \tag{B.7b}$$

These equations, after being inserted into the corresponding equations for $T_s(t)$ for the heating and cooling periods, respectively, together with equations (B.6) and (B.3a) give

$$\bar{T}_s - \bar{T}_m = \frac{1}{\sigma}\left\{\frac{H'\delta^2}{\alpha} + (H''-H')(1-\gamma)\left(\frac{\gamma\delta^2}{\sigma\alpha\tau_h}\right) \times \frac{\left[1-\exp\left(-\frac{\sigma\tau_h}{\gamma}\right)\right]\left[1-\exp\left(-\frac{\sigma\tau_c}{\gamma}\right)\right]}{\left[1-\exp\left(-\frac{\sigma\tau_h}{\gamma}-\frac{\sigma\tau_c}{\gamma}\right)\right]} - \frac{H'\delta^2\gamma}{\sigma\alpha}\right\} \tag{B.8}$$

Now,

$$H' = -\frac{F_0'}{\rho C_s l} \tag{B.9a}$$

$$H'' - H' = \frac{F_0'}{\rho C_s l}\left(1+\frac{\tau_h}{\tau_c}\right) \tag{B.9b}$$

$$\bar{T}_a - \bar{T}_s = -\frac{F_0'}{h'}. \tag{B.9c}$$

Thus,

$$\bar{T}_s - \bar{T}_m = (\bar{T}_a - \bar{T}_s)\left(\frac{h'}{\sigma l k}\right)\left\{\delta^2(1-\gamma) - (1-\gamma)\left(\frac{\delta^2\gamma}{\sigma}\right)\left(\frac{1}{\tau_c}+\frac{1}{\tau_h}\right) \times \frac{\left[1-\exp\left(-\frac{\sigma\tau_h}{\gamma}\right)\right]\left[1-\exp\left(-\frac{\sigma\tau_c}{\gamma}\right)\right]}{\left[1-\exp\left(-\frac{\sigma\tau_h}{\gamma}-\frac{\sigma\tau_c}{\gamma}\right)\right]}\right\} \tag{B.10}$$

Therefore,

$$\frac{1}{h^{*\prime}} = \frac{1}{h'}\left[1 + \frac{h'\delta^2}{\sigma l k}\left\{1-\gamma-(1-\gamma)\frac{\gamma}{\sigma}\left(\frac{1}{\tau_c}+\frac{1}{\tau_h}\right)\left(1-\exp\left[-\frac{\sigma\tau_h}{\gamma}\right]\right) \times \left(1-\exp\left[-\frac{\sigma\tau_c}{\gamma}\right]\right)\left(1-\exp\left[-\frac{\sigma\tau_h}{\gamma}-\frac{\sigma\tau_c}{\gamma}\right]\right)\right\}. \tag{B.11}$$

For a hollow cylinder, this equation becomes

$$\frac{1}{h^{*\prime}} = \frac{1}{h'}\left[1 + \frac{h'\delta}{3k}\cdot\frac{6}{\sigma(\beta+1)}\left\{1-\gamma-(1-\gamma)\left(\frac{\gamma}{\sigma}\right)\left(\frac{1}{\tau_c}+\frac{1}{\tau_h}\right)\left(1-\exp\left[-\frac{\sigma\tau_h}{\gamma}\right]\right) \left(1-\exp\left[-\frac{\sigma\tau_c}{\gamma}\right]\right)\Big/\left(1-\exp\left[-\frac{\sigma\tau_h}{\gamma}-\frac{\sigma\tau_c}{\gamma}\right]\right)\right\}\right]. \tag{B.12}$$

Thus,

$$\Phi = \frac{6}{\sigma(\beta+1)} \cdot \left[1 - \gamma - (1-\gamma)\left(\frac{\gamma}{\sigma}\right)\left(\frac{1}{\tau_c} + \frac{1}{\tau_h}\right)\left(1 - \exp\left[-\frac{\sigma\tau_h}{\gamma}\right]\right) \times \left(1 - \exp\left[-\frac{\sigma\tau_c}{\gamma}\right]\right) \Big/ \left(1 - \exp\left[-\frac{\sigma\tau_h}{\gamma} - \frac{\sigma\tau_c}{\gamma}\right]\right)\right]. \tag{B.13}$$

For very large periods

$$\lim_{\substack{\tau_c \to \infty \\ \tau_h \to \infty}} \Phi \to \frac{6(1-\gamma)}{\sigma(\beta+1)}. \tag{B.13a}$$

The solution of the original differential equation by the Laplace transformation method is

$$\underline{T_m} = [2 \cdot \underline{T_s} \cdot y(x)]/(\beta^2 - 1) \tag{B.14}$$

where

$$x \equiv r_1 \sqrt{\frac{s}{\alpha}} \tag{B.14a}$$

and

$$y(x) \equiv \frac{1}{x}\left[\frac{K_1(x)\,I_1(\beta x) - K_1(\beta x)\,I_1(x)}{K_1(\beta x)\,I_0(x) + K_0(x)\,I_1(\beta x)}\right]. \tag{B.14b}$$

Equation (B.14b) can be expanded in a series form to look

$$y(x) = 1 + \left[\frac{3\beta^4 - 4\beta^4 \ln\beta - 4\beta^2 + 1}{8(\beta^2 - 1)}\right]x^2 + \left[\frac{17\beta^4 - 13\beta^2 + 2}{96} - \frac{\beta^4 \ln\beta}{4} + \frac{\beta^6(2\ln\beta - 1)\ln\beta}{8(\beta^2 - 1)}\right]x^4 + \ldots. \tag{B.15}$$

Since by definition

$$\gamma = 1 + \lambda y_1'(0)$$

where

$$y_1'(0) = \frac{1}{(\beta - 1)^2}\, y'(0) \tag{B.16a}$$

and

$$\sigma = \lambda \tag{B.16b}$$

equation (B.13a) becomes

$$\Phi \to \frac{-6y_1'(0)}{(\beta + 1)} \tag{B.17}$$

where

$$y_1'(0) = \frac{3\beta^4 - 4\beta^4 \ln \beta - 4\beta^2 + 1}{8(\beta^2 - 1)(\beta - 1)^2}. \tag{B.17a}$$

Therefore, at the limit as $\Pi \to \infty$,

$$\lim_{\Pi \to \infty} \Phi \to \frac{3[\beta^4(4 \ln \beta - 3) + 4\beta^2 - 1]}{4(\beta - 1)^3 (\beta + 1)^2} \tag{B.18}$$

which is the same as the expression (5a) for $F(\beta)$.

Résumé—Un coefficient de transport de chaleur global a été établi pour des cylindres creux de longueur infinie chauffés et refroidis périodiquement. Ce coefficient de transport de chaleur corrigé comprend l'effet de la résistance de paroi et conserve la chaleur transportée par période. Le traitement a été généralisé en considérant que le chauffage ou le refroidissement a lieu à la surface intérieure ou extérieure du cylindre creux. Pour évaluer le coefficient de transport de chaleur global, toutes les températures moyennes temporelles nécessaires ont été déterminées theoriquement. Les expressions suivantes pour les résultats numériques ont été calculées sur un calculateur numérique IBM 7094. Les résultats sont présentés graphiquement sous la forme d'un facteur de correction pour le coefficient de transport de chaleur. Finalement, un exemple est présenté pour montrer l'application pratique des résultats.

Zusammenfassung—Ein Gesamtwärmeübergangskoeffizient wurde entwickelt für periodisch beheizte und gekühlte Hohlzylinder unendlicher Länge. Dieser korrigierte Wärmeübergangskoeffizient umfasst den Einfluss des Wandwiderstandes und berücksichtigt die pro Periode übertragene Wärmemenge. Die Behandlung wurde verallgemeinert durch die Annahme, dass die innere oder äussere Oberfläche des Hohlzylinders beheizt oder gekühlt wird. Zur Berechnung des Gesamtwärmeübergangskoeffizienten wurden alle zeitlichen Temperaturmittelwerte analytisch bestimmt. Die sich ergebenden numerischen Ausdrücke wurden auf einem 7094 Digitalrechner ausgewertet. Die Ergebnisse sind graphisch in Form eines Korrekturfaktors für den Wärmeübergangskoeffizienten angegeben. Um die praktische Anwendbarkeit der Ergebnisse zu zeigen wird ein Beispiel gerechnet.

Аннотация—Коэффициент теплообмена получен для периодически нагреваемых и последоваткльно охлаждаемых полых бесконечных цилиндров. Этот скорректированный коэффициент теплообмена включает эффект сопротивления стенки и тепло, передаваемое за период. Метод обобщается на нагрев или охлаждение на внутренней или на внешней поверхности полого цилиндра. Чтобы вычислить коэффициент теплообмена, необходимо аналитически определить все осредненные во времени температуры. Численные результаты были получены на вычислительной машине 7094. Результаты представлены графически в виде поправок к коэффициенту теплообмена. Иллюстрируется практическое применение результатов.